The Purchasing Power Parity Puzzle, Temporal Aggregation, and Half-Life Estimation

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Supplementary results

Derivation of (3) and (4): First observe that \( X_n = (1/M) \sum_{j=0}^{M-1} x_{Mn+j} \). Applying this operator to (1) yields

\[
X_n = \beta \frac{1}{M} \sum_{j=0}^{M-1} x_{Mn+j-1} + \frac{1}{M} \sum_{j=0}^{M-1} u_{Mn+j}.
\]  

(A1)

Using backward substitution in (1) gives \( x_t = \beta^{M-1} x_{t-M+1} + \sum_{k=0}^{M-2} \beta^k u_{t-k} \). Setting \( t = Mn + j - 1 \) in this expression and substituting the resulting expression into (A1) gives

\[
X_n = \beta \frac{1}{M} \sum_{j=0}^{M-1} \beta^{M-1} x_{Mn+j-M} + \frac{1}{M} \sum_{j=0}^{M-1} \sum_{k=0}^{M-2} \beta^k u_{Mn+j-k-1} + \frac{1}{M} \sum_{j=0}^{M-1} u_{Mn+j}
\]

\[
= \beta^M \frac{1}{M} \sum_{j=0}^{M-1} x_{Mn+j-M} + \frac{1}{M} \sum_{j=0}^{M-1} \left( u_{Mn+j} + \sum_{k=0}^{M-2} \beta^{k+1} u_{Mn+j-k-1} \right),
\]

which yields (3) and (4) directly.

\( \square \)

Derivation of (5) and (6): From (4) it may be shown that \( U_n \) has the representation

\[
U_n = \frac{1}{M} \sum_{l=-\infty}^{-1} \alpha_l u_{Mn+l} \text{ where } \alpha_l = \begin{cases} 
\sum_{m=|l|}^{M-1} \beta^m, & l < 0, \\
\sum_{m=l}^{M-1} \beta^{M-1-m}, & l \geq 0.
\end{cases}
\]  

(A2)

From (A2) and the white noise nature of \( u_t \),

\[
M^2 E \left( U_n^2 \right) = E \left( \sum_{k=-\infty}^{-1} \alpha_k u_{Mn+k} \right) \left( \sum_{l=-\infty}^{-1} \alpha_l u_{Mn+l} \right)
\]
From the definition of \( \dot{B} \) it follows that

\[
\text{plim } \dot{B} = B + \frac{\text{plim } N^{-1} \sum_n U_n X_{n-1}}{\text{plim } N^{-1} \sum_n X_n^2} = B + \frac{E(U_n X_{n-1})}{E(X_{n-1}^2)}.
\]

the expressions for the probability limits arising due to the stationary and ergodic nature of \( X_n \) under the assumptions of the model. Since \( X_n \) may be written \( X_n = \sum_{j=0}^{\infty} B^j U_{n-j} \) it follows that \( E(X_{n-1} U_n) = E \sum_j B^j U_{n-1-j} U_n = E(U_{n-1} U_n) = \omega_1 \) because \( U_n \) is MA(1).

Derivation of (7): From the definition of \( \dot{B} \) it follows that

\[
\text{plim } \dot{B} = B + \frac{\text{plim } N^{-1} \sum_n U_n X_{n-1}}{\text{plim } N^{-1} \sum_n X_n^2} = B + \frac{E(U_n X_{n-1})}{E(X_{n-1}^2)},
\]

the expressions for the probability limits arising due to the stationary and ergodic nature of \( X_n \) under the assumptions of the model. Since \( X_n \) may be written \( X_n = \sum_{j=0}^{\infty} B^j U_{n-j} \) it follows that \( E(X_{n-1} U_n) = E \sum_j B^j U_{n-1-j} U_n = E(U_{n-1} U_n) = \omega_1 \) because \( U_n \) is MA(1).
Turning to $E(X_{n-1}^2)$ it follows from (2) that

$$E\left(X_{n-1}^2 \right) = E\left( \frac{1}{M} \sum_{j=0}^{M-1} x_{Mn+j-M} \right)^2 = M^{-2} \sum_{j=0}^{M-1} \sum_{k=0}^{M-1} E\left(x_{Mn+j-M}x_{Mn+k-M} \right).$$

Since, for some $v > 0$, $x_t = \beta^v x_{t-v} + \sum_{j=0}^{v-1} \beta^j u_{t-j}$, it follows that

$$E(x_t x_{t-v}) = \beta^v E\left(x_{t-v}^2 \right) + \sum_{j=0}^{v-1} \beta^j E\left(u_{t-j} x_{t-v} \right) = \frac{\beta^v \sigma^2}{1 - \beta^2}$$

in view of $E(u_{t-j} x_{t-v}) = 0$ for $j < v$ and $E(x_{t-v}^2) = \sigma^2/(1 - \beta^2)$. Hence

$$M^2E\left(X_{n-1}^2 \right) = \frac{\sigma^2}{1 - \beta^2} \sum_{j=0}^{M-1} \sum_{k=0}^{M-1} \beta^{|j-k|} = \frac{\sigma^2}{1 - \beta^2} \left[ \frac{M(1 - \beta^2) - 2\beta(1 - \beta^M)}{(1 - \beta)^2} \right]. \quad (A4)$$

Combining (A3) and (A4) yields (7) as required. \qed